

OPEN MAPS HAVING THE BULA PROPERTY

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ABSTRACT. Every open continuous map f from a space X onto a paracompact C -space Y admits two disjoint closed sets $F_0, F_1 \subset X$, with $f(F_0) = Y = f(F_1)$, provided all fibers of f are infinite and C^* -embedded in X . Applications are demonstrated for the existence of “disjoint” usco multiselections of set-valued l.s.c. mappings defined on paracompact C -spaces, and for special type of factorizations of open continuous maps from metrizable spaces onto paracompact C -spaces. This settles several open questions raised in [13].

1. INTRODUCTION

All spaces in this paper are assumed to be at least completely regular. Following Kato and Levin [15], a continuous surjective map $f: X \rightarrow Y$ is said to have the *Bula property* if there exist two disjoint closed subsets F_0 and F_1 of X such that $f(F_0) = Y = f(F_1)$. In the sequel, such a pair (F_0, F_1) will be called a *Bula pair* for f . Bula [2] proved that every open continuous map f from a compact Hausdorff space onto a finite-dimensional metrizable space has this property provided all fibers of f are dense in themselves. This result was generalized in [9] to the case Y is countable-dimensional. Recently, Levin and Rogers [16] obtained a further generalization with Y being a C -space. The question whether the Levin-Rogers result remains true for open maps between metrizable spaces was raised in [13, Problem 1514] (if Y is not a C -space, this is not true, see [6] and [16]). Here, we provide a positive answer to this question:

Theorem 1.1. *Let X be a space, Y be a paracompact C -space, and let $f: X \rightarrow Y$ be an open continuous surjection such that all fibers of f are infinite and C^* -embedded in X . Then, f has the Bula property.*

The C -space property was originally defined by W. Haver [11] for compact metric spaces. Later on, Addis and Gresham [1] reformulated Haver’s definition for arbitrary spaces: A space X has property C (or X is a C -space) if for

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every sequence $\{\mathcal{W}_n : n = 1, 2, \dots\}$ of open covers of X there exists a sequence $\{\mathcal{V}_n : n = 1, 2, \dots\}$ of pairwise disjoint open families in X such that each \mathcal{V}_n refines \mathcal{W}_n and $\bigcup\{\mathcal{V}_n : n = 1, 2, \dots\}$ is a cover of X . It is well-known that every finite-dimensional paracompact space, as well as every countable-dimensional metrizable space, is a C -space [1], but there exists a compact metric C -space which is not countable-dimensional [24]. Let us also remark that a C -space X is paracompact if and only if it is countably paracompact and normal. Finally, let us recall that a subset $A \subset X$ is C^* -embedded in X if every bounded real-valued continuous function on A is continuously extendable to the whole of X .

Theorem 1.1 has several interesting applications. In Section 4, we apply this theorem to the graph of l.s.c. set-valued mappings defined on paracompact C -spaces and with point-images being closed and infinite subsets of completely metrizable spaces. Thus, we get that any such l.s.c. mapping has a pair of “disjoint”usco multiselections (see, Corollaries 4.3 and 4.4), which provides the complete affirmative solution to [13, Problem 1515] and sheds some light on [13, Problem 1516]. In this regard, let us stress the attention that, in Theorem 1.1, no restrictions on X are called a priori. In Section 5, we use Theorem 1.1 to demonstrate that every open continuous map f from a metric space (X, d) onto a paracompact C -space Y admits a special type of factorization ($Y \times [0, 1]$ throughout), provided all fibers of f are dense in themselves and complete with respect to d , see Theorem 5.1. This result is a common generalization of [9, Theorem 1.1] and [16, Theorem 1.2] (see, Corollary 5.10), and provides the complete affirmative solution to [13, Problem 1512].

Finally, a word should be said also for the proof of Theorem 1.1 itself. Briefly, a preparation for this is done in the next section. It is based on the existence of a continuous function $g : X \rightarrow [0, 1]$ such that g is not constant on each fiber $f^{-1}(y)$, $y \in Y$, of f (see, Theorem 2.1). Having already established this, the proof of Theorem 1.1 will be accomplished in Section 3 relying on a “parametric” version of an idea in the proof of [16, Theorem 1.3].

2. BULA PROPERTY AND FIBER-CONSTANT MAPS

Suppose that (F_0, F_1) is a Bula pair for a map $f : X \rightarrow Y$, where X is a normal space. Then, there exists a continuous function $g : X \rightarrow [0, 1]$ such that $g \upharpoonright f^{-1}(y)$ is not constant for every $y \in Y$. Indeed, take $g : X \rightarrow [0, 1]$ to be such that $F_i \subset g^{-1}(i)$, $i = 0, 1$. In this section, we demonstrate that the map f in Theorem 1.1 has this property as well. Namely, the following theorem will be proved.

Theorem 2.1. *Let X be a space, Y be a paracompact C -space, and let $f : X \rightarrow Y$ be an open continuous surjection such that all fibers of f are infinite and C^* -embedded in X . Then, there exists a continuous function $g : X \rightarrow [0, 1]$ such that $g \upharpoonright f^{-1}(y)$ is not constant for every $y \in Y$.*

To prepare for the proof of Theorem 2.1, let us recall some terminology. For spaces Y and Z , we will use $\Phi : Y \rightsquigarrow Z$ to denote that Φ is a set-valued mapping, i.e. a map from Y into the nonempty subsets of Z . A mapping $\Phi : Y \rightsquigarrow Z$ is *lower semi-continuous*, or l.s.c., if the set

$$\Phi^{-1}(U) = \{y \in Y : \Phi(y) \cap U \neq \emptyset\}$$

is open in Y for every open $U \subset Z$. A mapping $\Phi : Y \rightsquigarrow Z$ has an *open (closed)* graph if its graph

$$\text{Graph}(\Phi) = \{(y, z) \in Y \times Z : z \in \Phi(y)\}$$

is open (respectively, closed) in $Y \times Z$. A map $g : Y \rightarrow Z$ is a *selection* for $\Phi : Y \rightsquigarrow Z$ if $g(y) \in \Phi(y)$ for every $y \in Y$. Finally, let us recall that a space Z is C^m for some $m \geq 0$ if every continuous image of the k -dimensional sphere \mathbb{S}^k ($k \leq m$) in Z is contractible in Z .

In what follows, $\mathbb{I} = [0, 1]$ and $C(Z, \mathbb{I})$ denotes the set of all continuous functions from Z to \mathbb{I} . Also, $C(Z) = C(Z, \mathbb{R})$ is the set of all continuous functions on Z , and $C^*(Z)$ — that of all bounded members of $C(Z)$. As usual, $C^*(Z)$ is equipped with the *sup-metric* d defined by

$$d(g, h) = \sup \{|g(z) - h(z)| : z \in Z\}, \quad g, h \in C^*(Z).$$

It should be mentioned that $C^*(Z)$ is a Banach space, and $C(Z, \mathbb{I})$ is a closed convex subset of $C^*(Z)$. In the sequel, for $g \in C(Z, \mathbb{I})$ and $\varepsilon > 0$, we will use $B_\varepsilon^d(g)$ to denote the open ε -ball $B_\varepsilon^d(g) = \{h \in C(Z, \mathbb{I}) : d(g, h) < \varepsilon\}$.

The next statement is well-known and easy to prove.

Lemma 2.2. *Let X be a space, and let $A \subset X$ be a C^* -embedded subset of X . Then, the restriction map $\pi_A : C(X, \mathbb{I}) \rightarrow C(A, \mathbb{I})$ is an open continuous surjection.*

For a subset B of a space Z , let $\Theta_Z(B, \mathbb{I})$ be the set of all members of $C(Z, \mathbb{I})$ which are constant on B . If $B = Z$, then we will denote this set merely by $\Theta(Z, \mathbb{I})$. Note that $\Theta(Z, \mathbb{I})$ is, in fact, homeomorphic to \mathbb{I} .

Proposition 2.3. *Let X be a space, and let $A \subset X$ be an infinite C^* -embedded subset of X . Then, the set $C(X, \mathbb{I}) \setminus \Theta_X(A, \mathbb{I})$ is C^m for every $m \geq 0$.*

Proof. Consider the restriction map $\pi_A : C(X, \mathbb{I}) \rightarrow C(A, \mathbb{I})$, and take a continuous map $g : \mathbb{S}^n \rightarrow C(X, \mathbb{I}) \setminus \Theta_X(A, \mathbb{I})$ for some $n \geq 0$. Then, by Lemma 2.2, the composition $\pi_A \circ g : \mathbb{S}^n \rightarrow C(A, \mathbb{I}) \setminus \Theta(A, \mathbb{I})$ is also continuous. Observe that

$C(A, \mathbb{I})$ is an infinite-dimensional closed convex subset of $C^*(A)$ because A is infinite. From another hand, $\Theta(A, \mathbb{I})$ is one-dimensional being homeomorphic to \mathbb{I} . Then, by [20, Lemma 2.1], $C(A, \mathbb{I}) \setminus \Theta(A, \mathbb{I})$ is C^m for all $m \geq 0$. Hence, there exists a continuous extension $\ell : \mathbb{B}^{n+1} \rightarrow C(A, \mathbb{I}) \setminus \Theta(A, \mathbb{I})$ of $\pi_A \circ g$ over the $(n+1)$ -dimensional ball \mathbb{B}^{n+1} . Consider the set-valued mapping $\Phi : \mathbb{B}^{n+1} \rightsquigarrow C(X, \mathbb{I})$ defined by $\Phi(t) = \{g(t)\}$ if $t \in \mathbb{S}^n$ and $\Phi(t) = \pi_A^{-1}(\ell(t))$ otherwise. Since g is a selection for $\pi_A^{-1} \circ \ell \upharpoonright \mathbb{S}^n$ and, by Lemma 2.2, the restriction map is π_A is open, the mapping Φ is l.s.c. (see, [18, Examples 1.1* and 1.3*]). Also, Φ is closed and convex-valued in $C(X, \mathbb{I})$, hence in the Banach space $C^*(X)$ as well. Then, by the Michael's selection theorem [18, Theorem 3.2''], Φ has a continuous selection $h : \mathbb{B}^{n+1} \rightarrow C(X, \mathbb{I})$ which is, in fact, a continuous extension of g over \mathbb{B}^{n+1} . Moreover, $\pi_A(h(t)) = \ell(t) \notin \Theta(A, \mathbb{I})$ for all $t \in \mathbb{B}^{n+1}$, which completes the proof. \square

A function $\xi : X \rightarrow \mathbb{R}$ is *lower (upper) semi-continuous* if the set

$$\{x \in X : \xi(x) > r\} \quad (\text{respectively, } \{x \in X : \xi(x) < r\})$$

is open in X for every $r \in \mathbb{R}$. Suppose that $f : X \rightarrow Y$ is a surjective map. Then, to any $g : X \rightarrow \mathbb{I}$ we will associate the functions $\inf[g, f], \sup[g, f] : Y \rightarrow \mathbb{I}$ defined for $y \in Y$ by

$$\inf[g, f](y) = \inf \{g(x) : x \in f^{-1}(y)\},$$

and, respectively,

$$\sup[g, f](y) = \sup \{g(x) : x \in f^{-1}(y)\}.$$

Finally, we will also associate the function $\text{var}[g, f] : X \rightarrow \mathbb{I}$ defined by

$$\text{var}[g, f](y) = \sup[g, f](y) - \inf[g, f](y), \quad y \in Y.$$

Observe that $g : X \rightarrow \mathbb{I}$ is not constant on any fiber $f^{-1}(y)$, $y \in Y$, if and only if $\text{var}[g, f]$ is positive-valued. The following property is well-known, [12] (see, also, [7, 1.7.16]).

Proposition 2.4 ([12]). *Let X and Y be spaces, $f : X \rightarrow Y$ be an open surjective map, and let $g \in C(X, \mathbb{I})$. Then, $\sup[g, f]$ is lower semi-continuous, while $\inf[g, f]$ is upper semi-continuous. In particular, $\text{var}[g, f]$ is lower semi-continuous.*

We finalize the preparation for the proof of Theorem 2.1 with the following proposition.

Proposition 2.5. *Let X and Y be spaces, and let $f : X \rightarrow Y$ be an open surjective map. Then the set-valued mapping $\Theta : Y \rightsquigarrow C(X, \mathbb{I})$ defined by $\Theta(y) = \Theta_X(f^{-1}(y), \mathbb{I})$, $y \in Y$, has a closed graph.*

Proof. Take a point $y \in Y$ and $g \notin \Theta(y)$. Then, $\text{var}[g, f](y) > 2\delta$ for some positive number $\delta > 0$. By Proposition 2.4, there exists a neighbourhood V of y such that $\text{var}[g, f](z) > 2\delta$ for every $z \in V$. Then, $V \times B_\delta^d(g)$ is an open set in $Y \times C(X, \mathbb{I})$

such that $(V \times B_\delta^d(g)) \cap \text{Graph}(\Theta) = \emptyset$. Indeed, take $z \in V$ and $h \in B_\delta^d(g)$. Since $\text{var}[g, f](z) > 2\delta$, there are points $x, t \in f^{-1}(z)$ such that $|g(x) - g(t)| > 2\delta$. Since $h \in B_\delta^d(g)$, we have $|h(x) - g(x)| < \delta$ and $|h(t) - g(t)| < \delta$. Hence, $h(x) \neq h(t)$, which implies that $\text{var}[h, f](z) > 0$. Consequently, $h \notin \Theta(z)$. \square

Proof of Theorem 2.1. Consider the set-valued mapping $\Phi : Y \rightsquigarrow C(X, \mathbb{I})$ defined by $\Phi(y) = C(X, \mathbb{I}) \setminus \Theta(y)$, $y \in Y$, where Θ is as in Proposition 2.5. Then, by Proposition 2.5, Φ has an open graph, while, by Proposition 2.3, each $\Phi(y)$, $y \in Y$, is C^m for all $m \geq 0$. Since Y is a paracompact C -space, by the Uspenskij's selection theorem [26, Theorem 1.3], Φ has a continuous selection $\varphi : Y \rightarrow C(X, \mathbb{I})$. Define a map $g : X \rightarrow \mathbb{I}$ by $g(x) = [\varphi(f(x))](x)$, $x \in X$. Since f and φ are continuous, so is g (see, the proof of [10, Theorem 6.1]). Since $g \upharpoonright f^{-1}(y) = \varphi(y) \upharpoonright f^{-1}(y)$ and $\varphi(y) \notin \Theta(y)$ for every $y \in Y$, g is as required. \square

3. PROOF OF THEOREM 1.1

Suppose that X, Y and $f : X \rightarrow Y$ are as in Theorem 1.1. By Theorem 2.1, there exists a function $g \in C(X, \mathbb{I})$ such that $\inf[g, f](y) < \sup[g, f](y)$ for every $y \in Y$. Since $\inf[g, f]$ is upper semi-continuous and $\sup[g, f]$ is lower semi-continuous (by Proposition 2.4), and Y is paracompact, by a result of [3] (see, also, [5, 14]), there are continuous functions $\gamma_0, \gamma_1 : Y \rightarrow \mathbb{I}$ such that

$$\inf[g, f](y) < \gamma_0(y) < \gamma_1(y) < \sup[g, f](y), \quad y \in Y.$$

Let $\alpha_i = \gamma_i \circ f : X \rightarrow \mathbb{I}$, $i = 0, 1$. Then,

$$(3.1) \quad \inf[g, f](f(x)) < \alpha_0(x) < \alpha_1(x) < \sup[g, f](f(x)) \quad \text{for every } x \in X.$$

Next, define a continuous function $\ell : X \times \mathbb{I} \rightarrow \mathbb{R}$ by letting

$$\ell(x, t) = \frac{t - \alpha_0(x)}{\alpha_1(x) - \alpha_0(x)}, \quad (x, t) \in X \times \mathbb{I}.$$

Observe that $\ell(x, \alpha_0(x)) = 0$ and $\ell(x, \alpha_1(x)) = 1$ for every $x \in X$. Hence,

$$(3.2) \quad \ell(\{x\} \times [\alpha_0(x), \alpha_1(x)]) = [0, 1], \quad \text{for every } x \in X,$$

because ℓ is linear for every fixed $x \in X$. Finally, define a continuous function $h : X \rightarrow \mathbb{R}$ by $h(x) = \ell(x, g(x))$, $x \in X$. According to (3.1) and (3.2), we now have that, for every $y \in Y$,

$$h^{-1}((-\infty, 0]) \cap f^{-1}(y) \neq \emptyset \neq h^{-1}([1, +\infty)) \cap f^{-1}(y).$$

Then, $F_0 = h^{-1}((-\infty, 0])$ and $F_1 = h^{-1}([1, +\infty))$ are as required. The proof of Theorem 1.1 completes.

4. BULA PAIRS AND MULTISELECTIONS

A set-valued mapping $\varphi : Y \rightsquigarrow Z$ is called a *multiselection* for $\Phi : Y \rightsquigarrow Z$ if $\varphi(y) \subset \Phi(y)$ for every $y \in Y$. In this section, we present several applications of Theorem 1.1 about multiselections of l.s.c. mappings based on the following consequence of it.

Corollary 4.1. *Let Y be a paracompact C -space, Z be a normal space, and let $\Phi : Y \rightsquigarrow Z$ be an l.s.c. mapping such that each $\Phi(y)$, $y \in Y$, is infinite and closed in Z . Then, there exists a closed-graph mapping $\theta : Y \rightsquigarrow Z$ such that $\Phi(y) \cap \theta(y) \neq \emptyset \neq \Phi(y) \setminus \theta(y)$ for every $y \in Y$.*

Proof. Let $X = \text{Graph}(\Phi)$ be the graph of Φ , and let $f : X \rightarrow Y$ be the projection. Then, f is an open continuous map (because Φ is l.s.c.) such that all fibers of f are infinite. Let us observe that each $f^{-1}(y)$, $y \in Y$, is C^* -embedded in X . Indeed, take a point $y \in Y$, and a continuous function $g : f^{-1}(y) \rightarrow \mathbb{I}$. Since $f^{-1}(y) = \{y\} \times \Phi(y)$, we may consider the continuous function $g_0 : \Phi(y) \rightarrow \mathbb{I}$ defined by $g_0(z) = g(y, z)$, $z \in \Phi(y)$. Since Z is normal, there exists a continuous extension $h_0 : Z \rightarrow \mathbb{I}$ of g_0 . Finally, define $h : X \rightarrow \mathbb{I}$ by $h(t, z) = h_0(z)$ for every $t \in Y$ and $z \in \Phi(t)$. Then, h is a continuous extension of g . Thus, by Theorem 1.1, there are disjoint closed subsets $F_0, F_1 \subset X$ such that $f(F_0) = Y = f(F_1)$. Finally, take a closed set $F \subset Y \times Z$, with $F \cap X = F_0$, and define $\theta : Y \rightsquigarrow Z$ by $\text{Graph}(\theta) = F$. This θ is as required. \square

To prepare for our applications, we need also the following observation about l.s.c. multiselections of l.s.c. mappings.

Proposition 4.2. *Let Y be a paracompact space, Z be a space, $\Phi : Y \rightsquigarrow Z$ be an l.s.c. closed-valued mapping, and let $\Psi : Y \rightsquigarrow Z$ be an open-graph mapping, with $\Phi(y) \cap \Psi(y) \neq \emptyset$ for every $y \in Y$. Then, there exists a closed-valued l.s.c. mapping $\varphi : Y \rightsquigarrow Z$ such that $\varphi(y) \subset \Phi(y) \cap \Psi(y)$ for every $y \in Y$.*

Proof. Whenever $y \in Y$, there are open sets $V_y \subset Y$ and $W_y \subset Z$ such that $y \in V_y \subset \Phi^{-1}(W_y)$ and $V_y \times \overline{W_y} \subset \text{Graph}(\Psi)$. Indeed, take a point $z \in \Phi(y) \cap \Psi(y)$. Since Ψ has an open graph, there are open sets $O_y \subset Y$ and $W_y \subset Z$ such that $y \in O_y$, $z \in W_y$ and $O_y \times \overline{W_y} \subset \text{Graph}(\Psi)$. Then, $V_y = O_y \cap \Phi^{-1}(W_y)$ is as required. Now, for every $y \in Y$, define a closed-valued mapping $\varphi_y : V_y \rightsquigarrow Z$ by letting that $\varphi_y(t) = \overline{\Phi(t) \cap W_y}$, $t \in V_y$. According to [18, Propositions 2.3 and 2.4], each φ_y , $y \in Y$, is l.s.c. Next, using that Y is paracompact, take a locally-finite open cover \mathcal{U} of Y refining $\{V_y : y \in Y\}$ and a map $p : \mathcal{U} \rightarrow Y$ such that $U \subset V_{p(U)}$, $U \in \mathcal{U}$. Finally, define a mapping $\varphi : Y \rightsquigarrow Z$ by letting that

$$\varphi(y) = \bigcup \{ \varphi_{p(U)}(y) : U \in \mathcal{U} \text{ and } y \in U \}, \quad y \in Y.$$

This φ is as required. \square

In what follows, a mapping $\psi : Y \rightsquigarrow Z$ is *upper semi-continuous*, or u.s.c., if the set

$$\Phi^\#(U) = \{y \in Y : \Phi(y) \subset U\}$$

is open in Y for every open $U \subset Z$. Motivated by [19], we say that a pair (φ, ψ) of set-valued mapping $\varphi, \psi : Y \rightsquigarrow Z$ is a *Michael pair* for $\Phi : Y \rightsquigarrow Z$ if φ is compact-valued and l.s.c., ψ is compact-valued and u.s.c., and $\varphi(y) \subset \psi(y) \subset \Phi(y)$ for every $y \in Y$.

The following consequence provides the complete affirmative solution to [13, Problem 1515].

Corollary 4.3. *Let (Z, ρ) be metric space, Y be a paracompact C -space, and let $\Phi : Y \rightsquigarrow Z$ be an l.s.c. mapping such that each $\Phi(y)$, $y \in Y$, is infinite and ρ -complete. Then Φ has Michael pair $(\varphi, \psi) : Y \rightsquigarrow Z$ such that $\Phi(y) \setminus \psi(y) \neq \emptyset$ for every $y \in Y$.*

Proof. By Corollary 4.1, there is a closed-graph mapping $\theta : Y \rightsquigarrow Z$ such that $\Phi(y) \cap \theta(y) \neq \emptyset \neq \Phi(y) \setminus \theta(y)$ for every $y \in Y$. Consider the set-valued mapping $\Psi : Y \rightsquigarrow Z$ defined by $\text{Graph}(\Psi) = (Y \times Z) \setminus \text{Graph}(\theta)$. On one hand, by the properties of θ , we have that $\Phi(y) \cap \Psi(y) \neq \emptyset$ for every $y \in Y$. On another hand, Φ is closed-valued having ρ -complete values. Hence, by Proposition 4.2, there exists a closed-valued l.s.c. mapping $\Phi_0 : Y \rightsquigarrow Z$ such that $\Phi_0(y) \subset \Phi(y) \cap \Psi(y)$ for every $y \in Y$. Then, Φ_0 has also ρ -complete values and, by a result of [19], it has a Michael pair (φ, ψ) . This (φ, ψ) is as required. \square

We conclude this section with the following further application of Theorem 1.1 that sheds some light on [13, Problem 1516].

Corollary 4.4. *Let (Z, ρ) be metric space, Y be a paracompact C -space, and let $\Phi : Y \rightsquigarrow Z$ be an l.s.c. mapping such that each $\Phi(y)$, $y \in Y$, is infinite and ρ -complete. Then Φ has Michael pairs $(\varphi_i, \psi_i) : Y \rightsquigarrow Z$, $i = 0, 1$, such that $\psi_0(y) \cap \psi_1(y) = \emptyset$ for every $y \in Y$.*

Proof. According to Corollary 4.3, Φ has a Michael pair $(\varphi_0, \psi_0) : Y \rightsquigarrow Z$ such that $\Phi(y) \setminus \psi_0(y) \neq \emptyset$ for every $y \in Y$. Note that ψ_0 has a closed-graph being u.s.c. Then, just like in the proof of Corollary 4.3, there exists a Michael pair $(\varphi_1, \psi_1) : Y \rightsquigarrow Z$ for Φ such that $\psi_1(y) \subset \Phi(y) \setminus \psi_0(y)$, $y \in Y$. These (φ_i, ψ_i) , $i = 0, 1$, are as required. \square

5. OPEN MAPS LOOKING LIKE PROJECTIONS

Throughout this section, by a *dimension* of a space Z we mean the covering dimension $\dim(Z)$ of Z . In particular, Z is *0-dimensional* if $\dim(Z) = 0$.

We say that a continuous map $f : X \rightarrow Y$ has *dimension* $\leq k$ if all fibers of f have dimension $\leq k$. A continuous map $f : X \rightarrow Y$ is *light* if it is 0-dimensional,

i.e. if f has 0-dimensional fibers. Also, for convenience, we shall say that a map $f : X \rightarrow Y$ is *compact* if each fiber $f^{-1}(y)$, $y \in Y$, is a compact subset of X .

Suppose that $f : X \rightarrow Y$ is a surjective map. A subset $F \subset X$ will be called a *section* for f if $f(F) = Y$. In particular, we shall say that a section F for f is *open* (*closed*) if F is an open (respective, a closed) subset of X .

In this section, we demonstrate the following factorization theorem which is a partial generalization of [9, Theorem 1.1], also it provides the complete affirmative solution to [13, Problem 1512].

Theorem 5.1. *Let (X, d) be a metric space, Y be a paracompact C -space, and let $f : X \rightarrow Y$ be an open continuous surjection such that each fiber of f is dense in itself and d -complete. Also, let $U \subset X$ be an open section for f . Then, there exists a continuous surjective map $g : X \rightarrow Y \times \mathbb{I}$, a closed section $H \subset X$ for f , with $H \subset U$, and a copy $\mathfrak{C} \subset \mathbb{I}$ of the Cantor set such that*

- (a) $f = P_Y \circ g$, where $P_Y : Y \times \mathbb{I} \rightarrow Y$ is the projection, i.e. the following diagram is commutative.

$$\begin{array}{ccc} X & \xrightarrow{g} & Y \times \mathbb{I} \\ & \searrow f & \downarrow P_Y \\ & & Y \end{array}$$

- (b) $g(H) = Y \times \mathbb{I}$ and each $g^{-1}(y, c) \cap H$, $(y, c) \in Y \times \mathfrak{C}$, is compact and 0-dimensional.

In particular, $H_{\mathfrak{C}} = H \cap g^{-1}(Y \times \mathfrak{C})$ is a closed section for f such that $f|_{H_{\mathfrak{C}}}$ is a compact light map.

To prepare for the proof of Theorem 5.1, we introduce some terminology. For a metric space (X, d) , a nonempty subset $A \subset X$ and $\varepsilon > 0$, as in Section 2, we let

$$B_{\varepsilon}^d(A) = \{x \in X : d(x, A) < \varepsilon\}.$$

Also, we will use $\text{diam}_d(A)$ to denote the *diameter* of A with respect to d .

Following [9], to every nonempty subset $F \subset X$ we associate the number

$$\delta(F, X) = \inf \{1, \varepsilon : \varepsilon > 0 \text{ and } F \subset B_{\varepsilon}^d(S) \text{ for some nonempty finite } S \subset F\}.$$

In what follows, we let $\Omega(f)$ to be the set of all open sections for f . Also, for every $U \in \Omega(f)$, we introduce the d -mesh of U with respect to f by letting $\text{mesh}_d(U, f) = \sup \{\delta(f^{-1}(y) \cap U, X) : y \in Y\}$.

Proposition 5.2. *Let (X, d) be a metric space, Y be a paracompact C -space, and let $f : X \rightarrow Y$ be an open continuous surjection such that each fiber of f is dense in itself and d -complete. Then, for every $U \in \Omega(f)$ there are disjoint open sections $U_0, U_1 \in \Omega(f)$ such that $\overline{U_i} \subset U$, $i = 0, 1$.*

Proof. Consider U endowed with the compatible metric

$$\rho(x, y) = d(x, y) + \left| \frac{1}{d(x, X \setminus U)} - \frac{1}{d(y, X \setminus U)} \right|, \quad x, y \in U.$$

Next, define an l.s.c. mapping $\Phi : Y \rightsquigarrow X$ by $\Phi(y) = f^{-1}(y) \cap U$, $y \in Y$. Then, each $\Phi(y)$, $y \in Y$, is infinite and ρ -complete in U because each $f^{-1}(y)$, $y \in Y$, is dense in itself and d -complete. Hence, by Corollary 4.4, Φ has compact-valued u.s.c. multiselections $\psi_0, \psi_1 : Y \rightsquigarrow U$ such that $\psi_0(y) \cap \psi_1(y) = \emptyset$ for every $y \in Y$. In fact, ψ_0 and ψ_1 are compact-valued and u.s.c. as mappings from Y into the subsets of X . Hence, each $F_i = \bigcup \{\psi_i(y) : y \in Y\}$, $i = 0, 1$, is a closed subset of X , with $F_i \subset U$ and $f(F_i) = Y$. Since $F_0 \cap F_1 = \emptyset$, we can take disjoint open sets $U_0, U_1 \subset X$ such that $F_i \subset U_i \subset \overline{U_i} \subset U$, $i = 0, 1$. This completes the proof. \square

In our next considerations, to every nonempty subset F of a metric space (X, d) we associate (the possibly infinite) number

$$\text{td}_d(F) = \sup \{ \text{diam}_d(C) : C \subset F \text{ is connected} \}.$$

Next, for a surjective map $f : X \rightarrow Y$ and a section $U \in \Omega(f)$, we let

$$\text{td}_d(U, f) = \sup \{ \text{td}_d(f^{-1}(y) \cap U) : y \in Y \}.$$

In the proof of our next lemma and in the sequel, ω denotes the first infinite ordinal.

Lemma 5.3. *Let (X, d) be a metric space, Y be a paracompact C -space, and let $f : X \rightarrow Y$ be an open continuous surjection. Then, for every $\varepsilon > 0$, every $G \in \Omega(f)$ contains an $U \in \Omega(f)$, with $\text{mesh}_d(U, f) \leq \varepsilon$ and $\text{td}_d(U, f) \leq \varepsilon$.*

Proof. Let $\varepsilon > 0$ and $G \in \Omega(f)$. Whenever $y \in Y$ and $n < \omega$, take an open subset $W_y^n \subset G$ such that $y \in f(W_y^n)$ and $\text{diam}_d(W_y^n) < \varepsilon \cdot 2^{-(n+1)}$. Since f is open, each family $\mathcal{W}_n = \{f(W_y^n) : y \in Y\}$, $n < \omega$, is an open cover of Y . Since Y is a paracompact C -space, there now exists a sequence $\{\mathcal{V}_n : n < \omega\}$ of pairwise disjoint open families of Y such that each \mathcal{V}_n , $n < \omega$, refines \mathcal{W}_n and $\mathcal{V} = \bigcup \{\mathcal{V}_n : n < \omega\}$ is a locally-finite cover of Y . For convenience, for every $n < \omega$, define a map $p_n : \mathcal{V}_n \rightarrow Y$ by $V \subset f(W_{p_n(V)}^n)$, $V \in \mathcal{V}_n$, and next set $U_{p_n(V)} = f^{-1}(V) \cap W_{p_n(V)}^n$. We are going to show that

$$U = \bigcup \{U_{p_n(V)} : V \in \mathcal{V}_n \text{ and } n < \omega\}$$

is as required. Since \mathcal{V} is a cover of Y , U is a section for f , and clearly it is open. Take a point $y \in Y$, and set $\mathcal{V}_y = \{V \in \mathcal{V} : y \in V\}$. Then, \mathcal{V}_y is finite

and $|\mathcal{V}_y \cap \mathcal{V}_n| \leq 1$ for every $n < \omega$ (recall that each family \mathcal{V}_n , $n < \omega$, is pairwise disjoint). Hence, we can numerate the elements of \mathcal{V}_y as $\{V_k : k \in K(y)\}$ so that $V_k \in \mathcal{V}_k$, $k \in K(y)$, where $K(y) = \{n < \omega : \mathcal{V}_y \cap \mathcal{V}_n \neq \emptyset\}$. Next, set $U_k = U_{p_k(V_k)}$, $k \in K(y)$. Since

$$(5.1) \quad \text{diam}(U_k) < \varepsilon \cdot 2^{-(k+1)} \quad \text{for every } k \in K(y),$$

$f^{-1}(y) \cap U \subset B_\varepsilon^d(S)$ for every finite subset $S \subset f^{-1}(y) \cap U$, with $S \cap U_k \neq \emptyset$ for all $k \in K(y)$. Thus, $\delta(f^{-1}(y) \cap U, X) \leq \varepsilon$ which completes the verification that $\text{mesh}_d(U, f) \leq \varepsilon$.

To show that $\text{td}_d(U, f) \leq \varepsilon$, take a nonempty connected subset $C \subset f^{-1}(y) \cap U$, and points $x, z \in C$. Since C is connected and $C \subset \bigcup \{U_k : k \in K(y)\}$, there is a sequence k_1, \dots, k_m of distinct elements of $K(y)$ such that $x \in U_{k_1}$, $z \in U_{k_m}$ and $U_{k_i} \cap U_{k_j} \neq \emptyset$ if and only if $|i - j| \leq 1$, see [7, 6.3.1]. Therefore, by (5.1),

$$\begin{aligned} d(x, z) &\leq \sum_{i=1}^m \text{diam}_d(U_{k_i}) \leq \sum_{k \in K(y)} \text{diam}_d(U_k) \\ &< \sum_{k \in K(y)} \varepsilon \cdot 2^{-(k+1)} < \varepsilon \cdot \sum_{k=0}^{\infty} 2^{-(k+1)} = \varepsilon. \end{aligned}$$

Consequently, $\text{diam}_d(C) \leq \varepsilon$, which completes the proof. \square

Recall that a partially ordered set (T, \preceq) is called a *tree* if the set $\{s \in T : s \prec t\}$ is well-ordered for every point $t \in T$. Here, as usual, “ $s \prec t$ ” means that $s \preceq t$ and $s \neq t$. A *chain* η in a tree (T, \preceq) is a subset $\eta \subset T$ which is linearly ordered by \preceq . A maximal chain η in T is called a *branch* in T . Through this paper, we will use $\mathcal{B}(T)$ to denote the set of all branches in T . Following Nyikos [21], for every $t \in T$, we let

$$(5.2) \quad U(t) = \{\beta \in \mathcal{B}(T) : t \in \beta\},$$

and next we set $\mathcal{U}(T) = \{U(t) : t \in T\}$. It is well-known that $\mathcal{U}(T)$ is a base for a non-Archimedean topology on $\mathcal{B}(T)$, see [21, Theorem 2.10]. In fact, one can easily see that $s \prec t$ if and only if $U(t) \subset U(s)$, while s and t is incomparable if and only if $U(s) \cap U(t) = \emptyset$. In the sequel, we will refer to $\mathcal{B}(T)$ as a *branch space* if it is endowed with this topology.

For a tree (T, \preceq) , let $T(0)$ be the set of all minimal elements of T . Given an ordinal α , if $T(\beta)$ is defined for every $\beta < \alpha$, then we let

$$T \upharpoonright \alpha = \bigcup \{T(\beta) : \beta \in \alpha\},$$

and we will use $T(\alpha)$ to denote the minimal elements of $T \setminus (T \upharpoonright \alpha)$. The set $T(\alpha)$ is called the α^{th} -level of T . The *height* of T is the least ordinal α such that $T \upharpoonright \alpha = T$. In particular, we will say that T is an α -tree if its height is α . Finally,

we can also define the *height* of an element $t \in T$, denoted by $\text{ht}(t)$, which is the unique ordinal α such that $t \in T(\alpha)$.

In what follows, we will be mainly interested in ω -trees, and the following realization of the Cantor set as a branch space. Namely, let S be a set which has at least 2 distinct points, S^n be the set of all maps $t : n \rightarrow S$ (i.e., the n^{th} -power of S), and let

$$S^{<\omega} = \bigcup \{S^{n+1} : n < \omega\}.$$

Whenever $t \in S^{<\omega}$, let $\text{dom}(t)$ be the *domain* of t . Consider the partial order \preceq on $S^{<\omega}$ defined for $s, t \in S^{<\omega}$ by $s \preceq t$ if and only if

$$\text{dom}(s) \subset \text{dom}(t) \quad \text{and} \quad t \upharpoonright \text{dom}(s) = s.$$

Then, $(S^{<\omega}, \preceq)$ is an ω -tree such that its branch space $\mathcal{B}(S^{<\omega})$ is the Baire space S^ω . In particular, the branch space $\mathcal{B}(2^{<\omega})$ is the Cantor set 2^ω . In the sequel, we will refer to the tree $(2^{<\omega}, \preceq)$ as the *Cantor tree*.

By Proposition 5.2 and Lemma 5.3, using an induction on the levels of the Cantor tree $(2^{<\omega}, \preceq)$, we get the following immediate consequence.

Corollary 5.4. *Let $(2^{<\omega}, \preceq)$ be the Cantor tree, and let (X, d) , Y , $f : X \rightarrow Y$ and $U \in \Omega(f)$ be as in Theorem 5.1. Then, there exists a map $h : 2^{<\omega} \rightarrow \Omega(f)$ such that, for every two distinct members $s, t \in 2^{<\omega}$,*

- (a) $\overline{h(t)} \subset h(s) \subset U$ if $s \prec t$,
- (b) $h(s) \cap h(t) = \emptyset$ if s and t are incomparable,
- (b) $\text{mesh}_d(h(t), f) \leq 2^{-\text{ht}(t)}$ and $\text{td}_d(h(t), f) \leq 2^{-\text{ht}(t)}$.

We finalize the preparation for the proof of Theorem 5.1 with the following special case of it.

Lemma 5.5. *Let (X, d) , Y , $f : X \rightarrow Y$ and $U \in \Omega(f)$ be as in Theorem 5.1. Then, there exists a closed section $H \subset X$ of f , with $H \subset U$, and a surjective compact light map $\ell : H \rightarrow Y \times \mathfrak{C}$ such that $f \upharpoonright H = P_Y \circ \ell$, i.e. the following diagram is commutative.*

$$\begin{array}{ccc} H & \xrightarrow{\ell} & Y \times \mathfrak{C} \\ & \searrow f \upharpoonright H & \downarrow P_Y \\ & & Y \end{array}$$

In particular, $f \upharpoonright H$ is also a compact light map.

Proof. Let $h : 2^{<\omega} \rightarrow \Omega(f)$ be as in Corollary 5.4. Whenever $n < \omega$, consider the n^{th} -level of the Cantor tree $2^{<\omega}$, which is, in fact, 2^{n+1} . Then, set $H_n = h(2^{n+1})$, $n < \omega$, and $H = \bigcap \{H_n : n < \omega\}$. By (a) of Corollary 5.4, $\overline{H_{n+1}} \subset H_n \subset U$ for

every $n < \omega$ because each level of $2^{<\omega}$ is finite. Hence, H is a closed subset of X , with $H \subset U$. Let us see that H is a section for f . Indeed, take a point $y \in Y$, and a branch $\beta \in \mathcal{B}(2^{<\omega})$. Then, each $H_t(y) = h(t) \cap f^{-1}(y)$, $t \in \beta$, is a nonempty subset of $f^{-1}(y)$ (because $h(t) \in \Omega(f)$) such that $\overline{H_t(y)} \subset H_s(y)$ for $s \prec t$ (by (a) of Corollary 5.4) and $\lim_{t \in \beta} \delta(H_t(y), X) = 0$ (by (c) of Corollary 5.4). Hence, by [9, Lemma 3.2], $H_\beta(y) = \bigcap \{H_t(y) : t \in \beta\}$ is a nonempty compact subset of X . Clearly, $H_\beta(y) \subset H \cap f^{-1}(y)$ which completes the verification that H is a section for f . In fact, this defines a compact-valued mapping $\varphi : Y \times \mathcal{B}(2^{<\omega}) \rightsquigarrow H$ by letting $\varphi(y, \beta) = H_\beta(y) = \bigcap \{h(t) \cap f^{-1}(y) : t \in \beta\}$, $(y, \beta) \in Y \times \mathcal{B}(2^{<\omega})$. Since $\varphi(y, \beta) \subset f^{-1}(y)$, $(y, \beta) \in Y \times \mathcal{B}(2^{<\omega})$, the mapping φ is the inverse ℓ^{-1} of a surjective single-valued map $\ell : H \rightarrow Y \times \mathcal{B}(2^{<\omega})$. Also, $\ell(x) = (y, \beta)$ if and only if $x \in \varphi(y, \beta) \subset f^{-1}(y)$, hence $f \upharpoonright H = P_Y \circ \ell$.

To show that ℓ is continuous and light, take an open set $V \subset Y$, $t \in 2^{<\omega}$, and let $U(t)$ be as in (5.2). Then, $h(t)$ is an open set in X such that, by (b) of Corollary 5.4, $\ell^{-1}(y, \beta) = \varphi(y, \beta) \subset h(t)$ if and only if $t \in \beta$ (i.e., $\beta \in U(t)$). Consequently, $\ell^{-1}(V \times U(t)) = f^{-1}(V) \cap h(t) \cap H$ is open in H . Finally, take a nonempty connected subset $C \subset \ell^{-1}(y, \beta) = \varphi(y, \beta)$ for a point $y \in Y$ and a branch $\beta \in \mathcal{B}(2^{<\omega})$. Then, $C \subset h(t) \cap f^{-1}(y)$ for every $t \in \beta$ and therefore, by (c) of Corollary 5.4, $\text{diam}_d(C) = 0$. Hence, C is a singleton, which implies that $\ell^{-1}(y, \beta)$ is 0-dimensional being compact.

To show finally that $f \upharpoonright H$ is a compact light map, take a point $y \in Y$, and let us observe that $\ell \upharpoonright (f^{-1}(y) \cap H)$ is perfect. Indeed, take a branch $\beta \in \mathcal{B}(2^{<\omega})$ and a neighbourhood W of $\ell^{-1}(y, \beta)$ in X . Then, by [9, Lemma 3.2], there exists a $t \in \beta$, with $H_t(y) = h(t) \cap f^{-1}(y) \subset W$. In this case, $\ell^{-1}(y, \gamma) \subset W$ for every $\gamma \in U(t)$, where $U(t)$ is as in (5.2). Namely, $\gamma \in U(t)$ implies that $t \in \gamma$ and, therefore, $\ell^{-1}(y, \gamma) \subset H_t(y) \subset W$. Thus, $\ell \upharpoonright (f^{-1}(y) \cap H)$ is perfect, which implies that $f^{-1}(y) \cap H = \ell^{-1}(\{y\} \times \mathcal{B}(2^{<\omega}))$ is compact because so is $\mathcal{B}(2^{<\omega})$. Since $\mathcal{B}(2^{<\omega})$ is zero-dimensional and ℓ is a light map, according to the classical Hurewicz theorem (see, [8]), this also implies that $\dim(f^{-1}(y) \cap H) = 0$ which completes the proof. \square

Proof of Theorem 5.1. We repeat the arguments of [2, Theorem 1]. Briefly, let (X, d) , Y , $f : X \rightarrow Y$ and $U \in \Omega(f)$ be as in Theorem 5.1. By Lemma 5.5, there exists a closed section $H \subset X$ of f , with $H \subset U$, and a continuous surjective map $\ell : H \rightarrow Y \times \mathfrak{C}$ such that $f \upharpoonright H$ is a compact light map, and $f \upharpoonright H = P_Y \circ \ell$. Take a continuous surjective map $\rho : \mathfrak{C} \rightarrow \mathbb{I}$ such that the set

$$D = \{t \in \mathbb{I} : |\rho^{-1}(t)| > 1\}$$

is countable. Also, let $P_{\mathfrak{C}} : Y \times \mathfrak{C} \rightarrow \mathfrak{C}$ be the projection. Then, using the Tietze-Urysohn theorem, extend $\rho \circ P_{\mathfrak{C}} \circ \ell$ to a continuous map $u : X \rightarrow \mathbb{I}$. In this way,

we have that

$$u(f^{-1}(y) \cap H) = \mathbb{I} \quad \text{for every } y \in Y.$$

Then, we can define our $g : X \rightarrow Y \times \mathbb{I}$ by $g(x) = (f(x), u(x))$, $x \in X$. As for the second part of Theorem 5.1, take a copy \mathfrak{C} of the Cantor set in $\mathbb{I} \setminus D$, which is possible because D is countable. Then, by the properties of ρ , we have that $g^{-1}(Y \times \{c\}) \cap H = \ell^{-1}(Y \times \{c\})$ for every $c \in \mathfrak{C}$. Hence, Lemma 5.5 completes the proof. \square

We finalize this paper with several applications of Theorem 5.1. In what follows, for a space X , let $\mathcal{F}(X)$ be the set of all nonempty closed subsets of X . Recall that the Vietoris topology τ_V on $\mathcal{F}(X)$ is generated by all collections of the form

$$\langle \mathcal{V} \rangle = \left\{ S \in \mathcal{F}(X) : S \subset \bigcup \mathcal{V} \text{ and } S \cap V \neq \emptyset, \text{ whenever } V \in \mathcal{V} \right\},$$

where \mathcal{V} runs over the finite families of open subsets of X . In the sequel, any subset $\mathcal{D} \subset \mathcal{F}(X)$ will carry the relative Vietoris topology τ_V as a subspace of $(\mathcal{F}(X), \tau_V)$. In fact, we will be mainly interested in the subset

$$\mathcal{F}(f) = \{H \in \mathcal{F}(X) : f(H) = Y\},$$

where $f : X \rightarrow Y$ is a surjective map.

Corollary 5.6. *Let (X, d) be a metric space, Y be a paracompact C -space, and let $f : X \rightarrow Y$ be an open continuous surjection such that each fiber of f is dense in itself and d -complete. Then, the set*

$$\mathcal{L}(f) = \{H \in \mathcal{F}(f) : f \upharpoonright H \text{ is a compact light map}\}$$

is dense in $\mathcal{F}(f)$ with respect to the Vietoris topology τ_V .

Proof. Take a closed section $F \in \mathcal{F}(f)$, and a finite family \mathcal{U} of open subsets of X , with $F \in \langle \mathcal{U} \rangle$. Then, $U = \bigcup \mathcal{U}$ is an open section for f , so, by Theorem 5.1, it contains a closed section $H \subset U$ such that $f \upharpoonright H$ is a compact light map. Take a finite set $S \in \langle \mathcal{U} \rangle$, and then set $Z = H \cup S$. Clearly, $Z \in \mathcal{L}(f) \cap \langle \mathcal{U} \rangle$, which completes the proof. \square

Proposition 5.7. *Whenever Y is a metrizable space, there exists a closed 0-dimensional subset $A \subset Y \times \mathfrak{C}$ such that $P_Y(A) = Y$.*

Proof. We follow the idea of [25, Lemma 4.1]. Fix a 0-dimensional metrizable space M and a perfect surjective map $h : M \rightarrow Y$. By [22, Proposition 9.1], there exists a continuous map $g : M \rightarrow Q$, where Q is the Hilbert cube, such that $h \Delta g : M \rightarrow Y \times Q$ is an embedding. Next, take a Milyutin map $p : \mathfrak{C} \rightarrow Q$, i.e. a surjective continuous map admitting an averaging operator between the function spaces $C(\mathfrak{C})$ and $C(Q)$, see [23]. According to [4], there exists a compact-valued lower semi-continuous map $\varphi : Q \rightsquigarrow \mathfrak{C}$ such that $\varphi(z) \subset p^{-1}(z)$ for all $z \in Q$. Applying Michael's 0-dimensional selection theorem [17], there exists a continuous

map $\ell : M \rightarrow \mathfrak{C}$, with $\ell(x) \in \varphi(g(x))$ for any $x \in M$. Then, $h\Delta\ell$ embeds M as a closed subset A of $Y \times \mathfrak{C}$. Obviously, A is 0-dimensional and $P_Y(A) = Y$. \square

Corollary 5.8. *Let X be a metrizable space, Y be a metrizable C -space, and let $f : X \rightarrow Y$ be an open perfect surjection such that each fiber of f is dense in itself. Then, the set*

$$\mathcal{F}_0(f) = \{H \in \mathcal{F}(f) : \dim(H) = 0\}$$

is dense in $\mathcal{F}(f)$ with respect to the Vietoris topology τ_V .

Proof. Take a closed section $F \in \mathcal{F}(f)$, and a finite family \mathcal{U} of open subsets of X , with $F \in \langle \mathcal{U} \rangle$. Then, $U = \bigcup \mathcal{U}$ is an open section for f , so, by Theorem 5.1, there exists a closed section H for f , with $H \subset U$, a continuous surjective map $g : X \rightarrow Y \times \mathbb{I}$, and a copy $\mathfrak{C} \subset \mathbb{I}$ of the Cantor set such that $f = P_Y \circ g$, $g(H) = Y \times \mathbb{I}$, and $f \upharpoonright (H \cap g^{-1}(Y \times \mathfrak{C}))$ is a light map. By Proposition 5.7, $Y \times \mathfrak{C}$ contains a closed 0-dimensional set A , with $P_Y(A) = Y$. Finally, take $B = H \cap g^{-1}(A)$ which is a closed section for f because $P_Y(A) = Y$. Since f is perfect, so is g . Hence, $g \upharpoonright B$ is a perfect light map and, according to the classical Hurewicz theorem, $\dim(B) = 0$. Then, $Z = B \cup S \in \mathcal{F}_0(f) \cap \langle \mathcal{U} \rangle$ for some (every) finite set $S \in \langle \mathcal{U} \rangle$. \square

To prepare for our last consequence, let us also observe the following property of 0-dimensional sections.

Proposition 5.9. *Let X be a compact metrizable space, and let $\mathcal{F}_0(X)$ be the subset of all 0-dimensional members of $\mathcal{F}(X)$. Then, $\mathcal{F}_0(X)$ is a G_δ -subset of $\mathcal{F}(X)$.*

Proof. Take a metric d on X compatible with the topology of X . Next, for every $H \in \mathcal{F}_0(X)$ and $n \geq 1$, take a pairwise disjoint finite family $\mathcal{V}_n(H)$ of open subsets of X such that $\text{diam}(V) < 1/n$, $V \in \mathcal{V}_n(H)$, and $H \in \langle \mathcal{V}_n(H) \rangle$. Then, each $\mathcal{V}_n = \bigcup \{ \mathcal{V}_n(H) : H \in \mathcal{F}_0(X) \}$, $n \geq 1$, is τ_V -neighbourhood of $\mathcal{F}_0(X)$, and clearly $\mathcal{F}_0(X) = \bigcap \{ \mathcal{V}_n(F) : n = 1, 2, \dots \}$. \square

According to Corollary 5.8 and Proposition 5.9, we have the following immediate consequence, which is the Levin-Rogers [16, Theorem 1.2].

Corollary 5.10 ([16]). *Let X be a compact metrizable space, Y be a metrizable C -space, and let $f : X \rightarrow Y$ be an open continuous surjection such that each fiber of f is dense in itself. Then, the set*

$$\mathcal{F}_0(f) = \{H \in \mathcal{F}(f) : \dim(H) = 0\}$$

is a dense G_δ -subset in $\mathcal{F}(f)$ with respect to the Vietoris topology τ_V .

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